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Oblique propagation of solitary electrostatic waves in multispecies plasmas

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Abstract
Analyses of the oblique propagation of large amplitude electrostatic waves, encountered in the literature, use as additional assumptions that the plasma response is quasi-neutral and that there is no more than one inertial fluid species, so that the result is either a second-order differential equation or a Sagdeev-type energy integral. A careful discussion shows that, as electrostatic modes require that plasma currents be weak, lest unwanted wave magnetic effects are generated, the nonlinear amplitudes, at significant obliquity, have to be small, or, for stronger nonlinearities, the obliquity must remain small. Within the frame of the multispecies model assumptions, the description always leads to a Sagdeev-type of integral, the derivation of which has been given in a general and compact notation, to avoid all possibility of algebraic oversights.

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1. Introduction

Oblique propagation of large amplitude electrostatic waves and structures has been studied in plasma models where one [1–10] species can be described as fluid (cold [2, 3, 5, 7, 8] or warm isothermal [1] or adiabatic [4, 6, 9, 10]), in the presence of one [2–5, 8] or more [1, 6, 7, 9, 10] inertialess constituents. Some of the latter are Boltzmann distributed [1–7, 9, 10] or have polytropic pressures [8]. Furthermore, sometimes beam species are included, which are treated as highly magnetized, so that only their equation of motion parallel to the static magnetic field needs to be considered [6, 9].

An unstated restriction of the models seems to be that there is no more than one inertial, magnetized fluid species, for good reasons, as the algebra in section 3 will show. This is because, contrary to what obtains for parallel propagation, at truly oblique propagation the equations of motion (per species) are not directly invertible nor integrable, so that the density is not expressible in terms of the electrostatic potential and there are not enough constants of the motion.

The analyses all use three essential assumptions: the waves are electrostatic, the plasma response is quasi-neutral and the nonlinear structures assume a stationary form in a suitably chosen co-moving frame. Provided there is no more than one fluid species, the resulting basic equation is either a second-order differential equation of the form [5, 10]

$$a(\varphi)\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}+b(\varphi)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2+c(\varphi)=0 \tag{1}$$

or a Sagdeev-type energy integral [1, 6, 9],

$$\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2+S(\varphi)=0, \tag{2}$$

in terms of either the (normalized) electrostatic potential φ of the nonlinear wave or structure, or, equivalently, of a density [2–4, 7] or velocity [8]. The co-moving coordinate is denoted here for brevity x. Furthermore, $a(\varphi), b(\varphi), c(\varphi)$ and $S(\varphi)$ are complicated functions of φ and the plasma compositional parameters, and it is not always obvious to verify on these expressions whether (1) might be integrated to obtain (2) or not. Yet it remains intriguing that two, apparently different, results might be obtained.

$S(\varphi)$ is usually called the Sagdeev pseudopotential, traditionally obtained as a function of φ, but it can equally well be expressed in another variable, such as a density [2–4, 7] or velocity [8]. The nonlinear structures have the characteristic potential hump or dip profiles of solitary waves [1–4, 6–9], unless they are driven by nonzero electric fields and show spiky behavior [5, 8, 10].

It is explicitly not the aim of the present paper to add, based on the same methodological assumptions, yet another plasma model to the existing literature, because the inherent complexity of the Sagdeev pseudopotential or equivalent descriptions then needs a fully numerical discussion, and these are often not physically illuminating. The papers quoted here [1–10] are not meant to constitute an exhaustive list but serve to illustrate a variety of models and physical problems.

Rather, precisely in view of the convoluted analytical treatment, when written out in full, it is prudent to have a closer look at the validity of the approximations, in particular to establish whether their joint application is self-consistent or what restrictions this might imply. This will be addressed in section 2, whereas in section 3 it will be shown that, within the frame of the model assumptions, the description always leads to a Sagdeev-type of integral, in other words, that (1) can be integrated to obtain (2). This will be written in a general notation, so as not to lose sight of the wood for the trees. Section 4 then briefly summarizes the conclusions.

2. Basic assumptions

Let us start from a multispecies plasma model with basic fluid equations of continuity and motion per species with index s,

$$\frac{\partial n_s}{\partial t}+\frac{\partial}{\partial x}(n_s v_{sx})=0, \tag{3}$$

$$\frac{\partial \mathbf{v}_s}{\partial t}+v_{sx}\frac{\partial \mathbf{v}_s}{\partial x}+\frac{1}{n_s m_s}\frac{\partial p_s}{\partial x}\mathbf{e}_x=\frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}_s\times\mathbf{B}). \tag{4}$$

These have been written for propagation along the x-axis, and n_s, p_s, q_s, m_s and $\mathbf{v}_s$ are the species density, pressure, (signed) charge, mass and fluid velocity, the last one in vector form. The electric field and magnetic induction are $\mathbf{E}$ and $\mathbf{B}$, respectively, and the latter has a static component $\mathbf{B}_0$, oriented for oblique propagation as $\mathbf{B}_0 = B_0(\mathbf{e}_x \cos\vartheta + \mathbf{e}_z \sin\vartheta)$, with ϑ being the angle between the directions of wave propagation and static field. Inertialess plasma constituents can easily be accommodated in this framework by taking the appropriate $m_s \to 0$ limit, plus possibly some other adaptations.

The first assumption, typical in the quest for large solitary modes, is that the nonlinear structures assume a stationary form ($\partial/\partial t = 0$) in a suitably chosen co-moving frame. In this frame the undisturbed plasma moves by at a velocity V along the x-axis, and hence the equation of continuity (3) can be integrated to yield the conservation of (mass) flux,

$$n_s v_{sx} = n_{s0} V. \tag{5}$$

The boundary conditions for solitary structures are such that $n_s \to n_{s0}$ and $v_{sx} \to V$, and later also $p_s \to p_{s0}$ and $\varphi \to 0$, far away from the nonlinear disturbance. For the other velocity components we will need $v_{sy} \to 0$ and $v_{sz} \to 0$.

On the other hand, the undisturbed form of the equations of motion (4) shows that we need

$$\mathbf{E}_0 + V\mathbf{e}_x \times \mathbf{B}_0 = \mathbf{0}, \tag{6}$$

so that there is a constant electric field [8] $E_{0y} = VB_0 \sin\vartheta$.

Multiplying the stationary form of the equations of motion (4) by $n_s m_s$ or scalarly by $n_s m_s \mathbf{v}_s$ and summing over all plasma species gives, with the help of (5), that

$$V\sum_s n_{s0} m_s \frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}x} + \sum_s \frac{\mathrm{d}p_s}{\mathrm{d}x}\,\mathbf{e}_x = \sum_s n_s q_s\,\mathbf{E} + \sum_s n_s q_s \mathbf{v}_s \times \mathbf{B}, \tag{7}$$

$$V\sum_s n_{s0} m_s \mathbf{v}_s \cdot \frac{\mathrm{d}\mathbf{v}_s}{\mathrm{d}x} + V\sum_s \frac{n_{s0}}{n_s}\frac{\mathrm{d}p_s}{\mathrm{d}x} = \sum_s n_s q_s \mathbf{v}_s \cdot \mathbf{E}. \tag{8}$$

Now the second essential assumption is brought in, namely the electrostatic approximation. Strictly speaking, for that to hold there can be no wave magnetic effects and hence from Ampère's law no currents. As a parenthesis, when electrostatic modes are discussed in a linearized description, this is deemed a valid approximation provided one is dealing with small wavelengths, namely large wave numbers [11]. However, when a Sagdeev-type pseudopotential treatment is given for large acoustic modes, one is, for their linear counterparts, actually in the small wave number domain, but can now have potentially large modes. The validity of the electrostatic description is therefore by no means obvious.

Combined with the other important hypothesis that there is quasi-neutrality, we have to put

$$\sum_s n_s q_s \mathbf{v}_s \simeq \mathbf{0}, \qquad \sum_s n_s q_s \simeq 0, \tag{9}$$

and the rhs of (7) and (8) vanish (approximately). The lhs of (7) can then be integrated to give global conservation of momentum, written in components as

$$\begin{aligned} &\sum_s n_{s0} m_s V(v_{sx} - V) + \sum_s (p_s - p_{s0}) \simeq 0, \\ &\sum_s n_{s0} m_s v_{sy} \simeq 0, \\ &\sum_s n_{s0} m_s v_{sz} \simeq 0. \end{aligned} \tag{10}$$

Similarly, the integration of the lhs of (8) is formally possible, provided some pressure–density relations exist per species, to be discussed in the following section. This yields

$$\frac{1}{2}\sum n_{s0}m_s\left(v_{sx}^2+v_{sy}^2+v_{sz}^2-V^2\right)+\sum n_{s0}\int_0^{\varphi}\frac{1}{n_s}\frac{\mathrm{d}p_s}{\mathrm{d}n_s}\frac{\mathrm{d}n_s}{\mathrm{d}\varphi}\,\mathrm{d}\varphi'\simeq 0. \tag{11}$$

As intimated already in section 1, more than one fluid species cannot be handled, because at truly oblique propagation the equations of motion are not invertible, in the sense that the density (per species) is not directly expressible in terms of the electrostatic potential. Therefore, if there is only one fluid species (with subscript f and inertial effects) whereas all other species are inertialess ($m_s = 0$ for all $s \neq f$), (11) indicates that $v_{fy} \simeq 0 \simeq v_{fz}$.

Let us backtrack for a moment and not use the quasi-neutrality assumption, but still consider electrostatic modes. In that case only the z component of (11) holds as written there. Consequently, $v_{fz} \simeq 0$, but now the z component of (4) for the one fluid species tells us that $v_{fy} \simeq 0$, unless the propagation is strictly perpendicular. The latter case needs a separate discussion, which can be gleaned from the treatment in the following section by putting there $\vartheta = \pi/2$.

In general, therefore, it is the electrostatic assumption which puts the strongest restrictions on the applicability of the model. Indeed, one of the other consequences for the one fluid species is that the y component of (4) then yields

$$(V - v_{fx})\sin\vartheta \simeq 0, \tag{12}$$

based on the presence of E_{0y} as defined in (6).

Consequently, if one adheres to the requirement that the currents are small for electrostatic modes, lest they generate unwanted wave magnetic effects, (12) indicates that the nonlinear amplitudes are weak, at serious obliquity, or for stronger nonlinearities the obliquity must remain small. This is also seen from the x component of (4), which in the case of $v_{fy} \simeq 0$ amounts to determining v_{fx} and n_f as functions of φ, as if we were at (almost) parallel propagation.

Unfortunately, these possible restrictions on the models quoted from the literature [1–10] are seldom discussed, and it is customary to use the electrostatic and possibly the quasi-neutrality approximations in plasma wave studies without further reference to Maxwell's equations and the restrictions they impose.

3. Reduction to a Sagdeev pseudopotential description

Supposing that we need not worry too much about the restrictions pointed out in the previous section but may use the electrostatic and quasi-neutrality assumptions at face value, we can prove that the description always leads, within this framework, to a Sagdeev-type of integral.

We now start from a model that contains one fluid species, meaning that for this species (3) and (4) are needed in full, while for all other constituents the density can ultimately be expressed in terms of the electrostatic potential. For inertialess species the main momentum balance is between the pressure and φ, which is easily invertible, provided the pressure–density relations are polytropic, in the sense that $p_s \propto n_s^{\gamma_s}$, with index γ_s. Other species, like highly magnetized cold beams [9], are essentially treated as if the propagation were parallel. The one fluid species will also be assumed to have a polytropic pressure–density relation. All polytropic relations include isothermal ($\gamma_s = 1$) and adiabatic ($\gamma_s = 3$) pressure variations as special cases.

Invoking then quasi-neutrality means that one can express the density of the fluid component of the plasma as

$$n_f(\varphi) \simeq -\frac{1}{q_f}\sum_{s\neq f} n_s(\varphi)q_s, \tag{13}$$

as is done in some of the papers [5, 9, 10], unless there are only two species and the treatment is using their common density [2–4] or velocity component along the propagation direction [8] as the unknown variable. Yet another possibility is to use the density of one of the Boltzmann species to replace φ [7]. These descriptions, of course, are fundamentally equivalent to what is written in (13).

If one were to relax the quasi-neutrality assumption and use Poisson's equation in full,

$$n_f = -\frac{1}{q_f}\sum_{s\neq f} n_s(\varphi)q_s - \frac{\varepsilon_0}{q_f}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}, \tag{14}$$

we note that n_f is no longer a function of φ alone, but also depends on $\mathrm{d}^2\varphi/\mathrm{d}x^2$. This will preclude some of the integrations needed to arrive at a single differential equation or at an energy integral.

Even though the inertialess or beam density functions $n_s(\varphi)$ $(s \neq f)$ might be complicated, (13) means that we can know $p_f \propto n_f^{\gamma_f}$ and, through (5), also v_{fx} as functions of φ. We will, for economy of notation, continue in the following in using n_f as a shorthand for the rhs of (13) and omit on this and related functions the explicit φ dependence, all in the hope of avoiding getting lost in algebraic complications that obscure the line of reasoning.

Writing the components of the stationary form of (4) for the sole fluid component gives

$$v_{fx}\frac{\mathrm{d}v_{fx}}{\mathrm{d}x} + \frac{1}{n_f m_f}\frac{\mathrm{d}p_f}{\mathrm{d}x} + \frac{q_f}{m_f}\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \Omega_f v_{fy}\sin\vartheta, \tag{15}$$

$$v_{fx}\frac{\mathrm{d}v_{fy}}{\mathrm{d}x} = \Omega_f[(V - v_{fx})\sin\vartheta + v_{fz}\cos\vartheta], \tag{16}$$

$$v_{fx}\frac{\mathrm{d}v_{fz}}{\mathrm{d}x} = -\Omega_f v_{fy}\cos\vartheta, \tag{17}$$

where we have introduced the (signed) gyrofrequency $\Omega_f = q_f B_0/m_f$. Before going on, we note that for parallel propagation ($\vartheta = 0$) (15) becomes decoupled from (16) and (17), and the latter then can be combined to give $v_{fy} = v_{fz} = 0$. For perpendicular propagation ($\vartheta = \pi/2$) it is (17) which is decoupled, leading to $v_{fz} = 0$, and (15) and (16) can be combined as explained below for the general case, by simply substituting $\vartheta = \pi/2$ in the relevant steps.

Elimination of v_{fy} between (15) and (17) gives, after multiplication by n_f, that

$$\frac{\mathrm{d}v_{fx}}{\mathrm{d}x}\cos\vartheta + \frac{\mathrm{d}v_{fz}}{\mathrm{d}x}\sin\vartheta + \frac{1}{n_{f0}m_f V}\frac{\mathrm{d}p_f}{\mathrm{d}x}\cos\vartheta + \frac{q_f}{n_{f0}m_f V}n_f\frac{\mathrm{d}\varphi}{\mathrm{d}x}\cos\vartheta = 0. \tag{18}$$

This can be integrated to

$$(v_{fx} - V)\cos\vartheta + v_{fz}\sin\vartheta + \frac{c_{tf}^2\cos\vartheta}{\gamma_f V}\left[\left(\frac{n_f}{n_{f0}}\right)^{\gamma_f} - 1\right] + \frac{q_f\cos\vartheta}{n_{f0}m_f V}\int_0^{\varphi} n_f\,\mathrm{d}\varphi' = 0, \tag{19}$$

where the thermal velocity c_{tf} of the fluid species is defined through $c_{tf}^2 = \gamma_f p_{f0}/n_{f0}m_f$. Other boundary conditions can easily be accommodated in this and subsequent integrations and do not detract from the general line of thought.

Using (5) to express v_{fx} in terms of n_f, and the polytropic relation $p_f \propto n_f^{\gamma_f}$ together with the definition of c_{tf} allows a rewrite of the lhs of (15), for brevity, as

$$v_{fx}\frac{\mathrm{d}v_{fx}}{\mathrm{d}x} + \frac{1}{n_f m_f}\frac{\mathrm{d}p_f}{\mathrm{d}x} + \frac{q_f}{m_f}\frac{\mathrm{d}\varphi}{\mathrm{d}x} = F\frac{\mathrm{d}\varphi}{\mathrm{d}x}, \tag{20}$$

where

$$F = \left[c_{tf}^2\left(\frac{n_f}{n_{f0}}\right)^{\gamma_f-2} - V^2\left(\frac{n_{f0}}{n_f}\right)^3\right]\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{n_f}{n_{f0}}\right) + \frac{q_f}{m_f} \tag{21}$$

is, through n_f, also a function of φ. Note that (20) could equally well be written as

$$F\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{c_{tf}^2}{\gamma_f - 1}\left(\frac{n_f}{n_{f0}}\right)^{\gamma_f-1} + \frac{V^2}{2}\left(\frac{n_{f0}}{n_f}\right)^2 + \frac{q_f}{m_f}\varphi\right], \tag{22}$$

but ultimately the algebra is equivalent to that given below. All this allows us to recast the derivative of (15) as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right) = \Omega_f\frac{\mathrm{d}v_{fy}}{\mathrm{d}x}\sin\vartheta. \tag{23}$$

With the help of (16) this gives

$$v_{fx}\frac{\mathrm{d}}{\mathrm{d}x}\left(F\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right) = \Omega_f^2[(V - v_{fx})\sin\vartheta + v_{fz}\cos\vartheta]\sin\vartheta. \tag{24}$$

It is now possible to eliminate v_{fz} between this equation and (19) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right) + G = 0, \tag{25}$$

in terms of a new function G with a more complicated φ dependence,

$$G = \frac{\Omega_f^2 n_f \cos^2\vartheta}{n_{f0}V^2}\left\{\frac{c_{tf}^2}{\gamma_f}\left[\left(\frac{n_f}{n_{f0}}\right)^{\gamma_f} - 1\right] + \frac{q_f}{m_f}\int_0^{\varphi}\frac{n_f}{n_{f0}}\mathrm{d}\varphi'\right\} + \Omega_f^2\left(1 - \frac{n_f}{n_{f0}}\right). \tag{26}$$

After multiplying (25) by $F\mathrm{d}\varphi/\mathrm{d}x$ the resulting expression can easily be integrated and a Sagdeev-type energy integral (2) obtains, with

$$S(\varphi) = \frac{1}{F(\varphi)^2}\int_0^{\varphi} F(\varphi')G(\varphi')\,\mathrm{d}\varphi', \tag{27}$$

where the φ dependence has been restored. On the other hand, when (25) is worked out, we get

$$F(\varphi)\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + \frac{\mathrm{d}F(\varphi)}{\mathrm{d}\varphi}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 + G(\varphi) = 0, \tag{28}$$

showing that the link between (1) and (2) implies that

$$\frac{F(\varphi)}{a(\varphi)} = \frac{\mathrm{d}F(\varphi)/\mathrm{d}\varphi}{b(\varphi)} = \frac{G(\varphi)}{c(\varphi)} = \rho(\varphi). \tag{29}$$

Here $\rho(\varphi)$ is a possible integrating factor, and (29) leads to a relation between $a(\varphi)$ and $b(\varphi)$, involving $\rho(\varphi)$,

$$b(\varphi) = \frac{\mathrm{d}a(\varphi)}{\mathrm{d}\varphi} + a(\varphi)\frac{\mathrm{d}\ln\rho(\varphi)}{\mathrm{d}\varphi}. \tag{30}$$

The steps leading to (30) are basically very simple: eliminate v_{fy} between (15) and (17) to obtain (19), define F as in (21) and take the derivative of (15) to arrive at (25), and hence (28), by eliminating v_{fz} between (19) and (24). This reasoning is generic and does not depend on the explicit expressions of $F(\varphi)$ and $G(\varphi)$.

Although (30) has not been mentioned nor used in the papers [5, 10] having (1) as the starting point for their discussions, it is obeyed in the paper by Reddy *et al* [5] without the need for an integrating factor ($\rho = 1$). In the paper by Maharaj *et al* [10], the integrating factor essentially corresponds to (13) for the (fluid) dust component of the plasma ($\rho \propto n_f$), up to changes of notation and normalization. Missing the integrating factor is easy in complicated derivations of the sort, if all intermediate details are written out in full and expressions get unwieldy.

4. Conclusions

Analyses of the oblique propagation of large amplitude electrostatic waves, as encountered in the literature, use three essential assumptions, that the waves are electrostatic, the plasma response is quasi-neutral and the nonlinear structures assume a stationary form in a co-moving frame. Consequently, the resulting basic equation is either a second-order differential equation or a Sagdeev-type energy integral.

In this paper, rather than working out a detailed model, the joint validity of the approximations has been investigated, to establish whether they are self-consistent and what restrictions are implied. The existence of electrostatic modes requires that plasma currents be sufficiently weak, lest they generate wave magnetic effects. It then follows that either, at significantly oblique propagation, the nonlinearities are weak, or for stronger nonlinearities, the angle of propagation to the magnetic field must remain relatively small.

Next, it has been shown that, within the frame of the model assumptions, the description always leads to a Sagdeev-type of integral. However, the methods only work provided there is no more than one inertial fluid species, otherwise the set of equations cannot be reduced to one single equation. The discussion has been given in as compact a notation as possible, to clearly bring out the line of thought and to avoid algebraic oversights.

It is hoped that future work will keep the basic restrictions on amplitudes and obliquity in mind, specially when discussing large-scale astrophysical applications.

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